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A General Framework for Detectability in Stochastic Discrete-Event Systems

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***Abstract*—This letter proposes a new framework to capture detectability property in stochastic discrete-event systems. A new notion, name *Partition-based Detectability* or *P-Detectability*, is proposed based on partitions of the system state space, rather than the state space itself. In other words, the proposed P-Detectability focuses on the system capability to detect certain state group from other state groups, while ignoring the ambiguity between individual states within the same state group. As a consequence, the proposed P-Detectability allows users to define customized public and cover to ignore irrelevant ambiguity. Compared to existing notions such as A-Detectability and A-Diagnosabiltiy, the proposed notion is shown to be more general. A necessary and sufficient condition to verify P-Detectability, together with a testing algorithm, are developed.**

***Index Terms*—Discrete event systems, stochastic systems, observability, detectability, state estimation.**

I. Introduction

DISCRETE-EVENT systems (DESs) have been widely used to model and analyze high level behaviors of complex engineering systems [1], [2], [3]. As DESs are usually only partially observable, various observational properties have been proposed and studied in literature [3], [4], [5], [6], [7]. In particular, the property of *fault diagnosability* and *state detectability* specify the system property that certain information should be revealed to an external monitor or supervisor for analysis and control purpose [8]. For example, to ensure system operations, failure diagnosability requires that every fault can eventually be detected, which was first studied under the notion of *Diagnosability* [5], [9] and later extended to distributed DESs [10], [11], decentralized DESs [12], [13], and stochastic DESs [4], [14]. Specifically, the notion of *A-Diagnosability* is proposed in [4], which requires that all fault events can be detected with arbitrary tolerable miss detection rate as long as the system has been observed for sufficiently long period. Fault diagnosability is also studied in the context of DESs with temporal logic specification in [15], [16].

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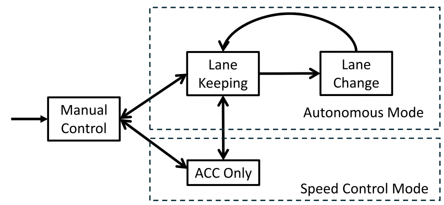

Fig. 1. An autonomous vehicle example.

The notion of diagnosability is later generalized to the notion of *detectability*, by focusing on any arbitrary states rather than fault states only [17], [18], [19], [20], [21], [22]. For example, *initial-state detectability* studied in [17] requires the initial state to be detectable with probability larger than a predefined threshold, while *A-Detectability* studied in [21] requires that the probability of uniquely detecting the current system state surely converges to 1 as the system evolves. Authors in [22] study the notions of detectability and diagnosability under the same framework, by reducing each of them as an instance of HyperLTL model checking problem.

One limitation of these prior works on detectability and diagnosability is that they are all comparing certain system behaviors against the rest of possible system behaviors. For example, in the case of detectability, the goal is to distinguish each of the system state from the rest of all system states. However, in practice, the ambiguity of certain system behaviors may not be of interests and can be tolerated, while the ambiguity of other pairs of system behaviors should be distinguished. Consider an autonomous vehicle example shown in Fig. 1. The vehicle can be in several states, including "Manual Control" (no autonomy), "Lane Keeping" (full autonomy), "Lane Change" (full autonomy), and "ACC only" (partial autonomy – only longitudinal speed control is automated, while the driver controls lateral movement). For the purpose of driver awareness monitoring, one may wish to detect if the vehicle is in "ACC Only", "Lane Keeping", or "Lane Change", provided that the vehicle is in autonomous mode or speed control mode. In other words, any ambiguity with "Manual Control" is of no interests and can be ignored when verifying detectability. In another example, suppose the desired system property require that the driver should be aware of (or "detect") whether the vehicle is in autonomous mode (either "Lane

Keeping" or "Lane Change"), speed control mode, or manual control mode, and it is acceptable if "Lane Keeping" and "Lane Change" are not detectable from each other. In the first example, the detectability is defined with respect to a subset of system states (rather than the whole state space), while in the second example, the detectability is defined between state groups (rather than state itself). Unfortunately, existing notions on detectability or diagnosability cannot capture such property described above.

To address this issue, this letter proposes a new notion, namely *Partition-based Detectability* or *P-Detectability*, for stochastic DESs. The proposed P-Detectability partitions the system state space into multiple groups, so that the ambiguity within each group is acceptable. In addition, a subset of these state groups is selected as "public" while another subset of these state groups is selected as "cover". Then instead of requiring each individual state to be detectable, P-Detectability requires that any public state group can be surely detected from the cover state groups. Therefore, the ambiguity within a public state group is ignored, and so is the ambiguity with a state group that is not a cover. Comparisons to A-Detectability and A-Diagnosabiltiy are discussed. Particularly, it is shown that both A-Diagnosability and A-Detectability can be reduced to P-Detectabiltiy, making the proposed notion more general. A necessary and sufficient condition to verify the proposed P-Detectability is provided, together with a testing algorithm. Unfortunately, as P-Detectability is more general than A-Diagnosability and A-Detectability, its verification problem is PSPACE-Hard.

The remainder of this letter is organized as follows. Section II provides necessary notations and preliminaries. Section III proposes P-Detectability for stochastic DESs and compares it to existing notions of A-Detectability and A-Diagnosability. Section IV provides a necessary and sufficient condition for verifying P-Detectability and a testing algorithm, while this letter is concluded in Section V.

II. Notation and Preliminary

Given an event set Σ, define $\overline{\Sigma} := \Sigma \cup \{\epsilon\}$ where ϵ denotes "no-event". Denote Σ^* as the set of all finite sequence over Σ including ϵ. A *trace* is a member of Σ^* and a *language* is a subset of Σ^*. Denote $|s|$ as the length of s. Given $s \in \Sigma^*$ and $L \subseteq \Sigma^*$, let $L \backslash s := \{t \in \Sigma^* | st \in L\}$ be the set of traces in L after s.

We model a stochastic DES as a *stochastic automaton* $G = (X, \Sigma, \alpha, x_0)$, where X is the finite set of states, Σ is the finite set of events, $x_0 \in X$ is the initial state, and $\alpha : X \times \Sigma \times X \to [0, 1]$ is the transition probability function [23]. Note that α can be extended to domain $X \times \Sigma^* \times X$ in a natural way. We can then assign probability to $s \in \Sigma^*$ as $Pr(s) = \sum_{x \in X} \alpha(x_0, s, x)$. In addition, an automaton is said to be logical if $\alpha : X \times \Sigma \times X \to \{0, 1\}$, and a logical automaton is said to be deterministic if $\forall x \in X, \sigma \in \Sigma, \sum_{x' \in X} \alpha(x, \sigma, x') \in \{0, 1\}$. Define $L(G) := \{s \in \Sigma^* : \exists x \in X, \alpha(x_0, s, x) > 0\}$ as the language generated by G. Given a set of observable symbols Δ, the events can then be observed through an observation mask, $M : \overline{\Sigma} \to \overline{\Delta}$, where $\overline{\Delta} := \Delta \cup \{\epsilon\}$. An event σ is *unobservable* if $M(\sigma) = \epsilon$; the set of unobservable events is denoted by Σ_{uo} and the set of observable events is given by $\Sigma_o = \Sigma - \Sigma_{uo}$. The observation mask can be extended from domain Σ to Σ^* in a natural way.

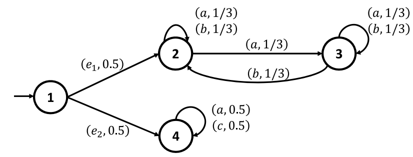

Fig. 2. A stochastic discrete-event system, with $M(e_1) = M(e_2) = \epsilon$, and $M(\sigma) = \sigma$ for $\sigma = a, b, c$.

For all $\sigma \in \Sigma$, define $\boldsymbol{\mu}(\sigma)$ as a $|X| \times |X|$ matrix, whose ijth element $\boldsymbol{\mu}_{ij}(\sigma) = \alpha(x_i, \sigma, x_j)$. Similarly, for $\delta \in \Delta$, define $\boldsymbol{\mu}_M(\delta)$ as a $|X| \times |X|$ matrix, whose ijth element is the probability of all traces originating at x_i, terminating at x_j and executing a sequence of unobservable events followed by a single observable event δ. Define $\boldsymbol{\lambda}$ as a $|X| \times |X|$ matrix, whose ijth element $\boldsymbol{\lambda}_{ij} = \sum_{\sigma \in \Sigma_{uo}} \boldsymbol{\mu}_{ij}(\sigma)$. Then we have,

$$\boldsymbol{\mu}_M(\delta) = \boldsymbol{\lambda} \boldsymbol{\mu}_M(\delta) + \sum_{\sigma \in \Sigma : M(\sigma) = \delta} \boldsymbol{\mu}(\sigma). \tag{1}$$

Solving (1) yields $\boldsymbol{\mu}_M(\delta)$. See [14], [24] for a similar example.

Given an automaton G, a *component* $C = (X_C, \alpha_C)$ of G is a "subgraph" of G such that $X_C \subseteq X$ and $\forall x, x' \in X_C$ and $\sigma \in \Sigma$, $\alpha_C(x, \sigma, x') = \alpha(x, \sigma, x')$. A component is a *strongly connected component* (SCC) or *irreducible* if $\forall x, x' \in X_C$, $\exists s \in \Sigma^*$ such that $\alpha_C(x, s, x') > 0$. An SCC C is *closed* if the probability of exiting C at each state is 0, i.e., $\forall x \in X_C$, $\sum_{\sigma \in \Sigma} \sum_{x' \in X_C} \alpha_C(x, \sigma, x') = 1$. In other words, if an SCC C is closed, then the probability of staying in C is 1 once entering it. A state is said to be *recurrent* if it belongs to a closed SCC.

Define $\pi(s) \in [0, 1]^{1 \times |X|}$ as the state distribution vector after execution of trace $s \in L(G)$, and $\tilde{\pi}(t) \in [0, 1]^{1 \times |X|}$ as the estimation of the state distribution after observing $t \in M(L(G))$. Note that both $\pi(s)$ and $\tilde{\pi}(t)$ are row vectors and can be recursively computed as follows. Given $s \in L(G)$ and $\sigma \in \Sigma$,

$$\pi(s\sigma) = \frac{\pi(s)\boldsymbol{\mu}(\sigma)}{||\pi(s)\boldsymbol{\mu}(\sigma)||}, \tag{2}$$

where $|| \cdot ||$ denotes 1-norm of a vector. Given $t \in M(L(G))$ and $\delta \in \Delta$,

$$\tilde{\pi}(t\delta) = \frac{\tilde{\pi}(t)\boldsymbol{\mu}_M(\delta)}{||\tilde{\pi}(t)\boldsymbol{\mu}_M(\delta)||}. \tag{3}$$

Example 1: Consider the system in Fig. 2, where $M(e_1) = M(e_2) = \epsilon$, and $M(\sigma) = \sigma$ for $\sigma = a, b, c$. The initial state is "1", i.e., $\pi(\epsilon) = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}$. It can be seen that

$$\boldsymbol{\mu}(e_1) = \begin{bmatrix} 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \boldsymbol{\mu}(e_2) = \begin{bmatrix} 0 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\boldsymbol{\mu}(a) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{1}{3} & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{3} & 0 \\ 0 & 0 & 0 & \frac{1}{2} \end{bmatrix}, \quad \boldsymbol{\mu}(b) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{3} & \frac{1}{3} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\boldsymbol{\mu}(c) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} \end{bmatrix}.$$

Therefore, after executing e_1aa, the state distribution becomes $\pi(e_1aa) = \begin{bmatrix} 0 & 1/3 & 2/3 & 0 \end{bmatrix}$. On the other hand, we have

$$\boldsymbol{\mu}_M(a) = \begin{bmatrix} 0 & \frac{1}{6} & \frac{1}{6} & \frac{1}{4} \\ 0 & \frac{1}{3} & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{3} & 0 \\ 0 & 0 & 0 & \frac{1}{2} \end{bmatrix}, \boldsymbol{\mu}_M(b) = \begin{bmatrix} 0 & \frac{1}{6} & 0 & \frac{1}{4} \\ 0 & \frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{3} & \frac{1}{3} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\boldsymbol{\mu}_M(c) = \begin{bmatrix} 0 & 0 & 0 & \frac{1}{4} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} \end{bmatrix}.$$

When observation $o = aa$ is observed, the state distribution is given by $\tilde{\pi}(o) = \begin{bmatrix} 0 & 4/21 & 8/21 & 9/21 \end{bmatrix}$. ■

III. Partition-Based Detectability for Stochastic DESs

A. State Partition

Let's partition the state space X into a set of K disjoint state group $X_1, \ldots, X_K$ such that: (1) $X_k \subseteq X$, $k = 1, \ldots, K$, (2) $X_k \cap X_j = \emptyset$ if $k \neq j$, and (3) $\bigcup_{k=1,\ldots,K} X_k = X$. Given $x \in X$, denote $\mathrm{id}(x)$ as the state group number that x belongs to, i.e., $x \in X_{\mathrm{id}(x)}$.

For each state group X_k, denote its membership index as $I_k \in \{0,1\}^{|X|}$ such that the nth element of I_k equals 1 if $x_n \in X_k$ and otherwise it is 0. For $\mathcal{K} = \{k_1, \ldots, k_{|\mathcal{K}|}\} \subseteq \{1, \ldots, K\}$, define

$$I_{\mathcal{K}} = \begin{bmatrix} I_{k_1} & I_{k_2} & \cdots & I_{k_{|\mathcal{K}|}} \end{bmatrix} \in \{0,1\}^{|X| \times |\mathcal{K}|}. \tag{4}$$

In other words, $\mathcal{K}$ here denotes a subset of state groups and $I_{\mathcal{K}}$ is the index matrix for $\mathcal{K}$ with each column corresponding to the membership index for one element in $\mathcal{K}$.

Example 2: Consider the system in Fig. 2. Let $X_1 = \{1\}$, $X_2 = \{2,3\}$, and $X_3 = \{4\}$. Then, the membership index for each state group is given by $I_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}^T$, $I_2 = \begin{bmatrix} 0 & 1 & 1 & 0 \end{bmatrix}^T$, and $I_3 = \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix}^T$. Let $\mathcal{K}_1 = \{1,3\}$ and $\mathcal{K}_2 = \{2\}$, then we have

$$I_{\mathcal{K}_1} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}, \qquad I_{\mathcal{K}_2} = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}.$$

According to Example 1, after executing $s = e_1aa$, the state distribution is $\pi(s) = \begin{bmatrix} 0 & 1/3 & 2/3 & 0 \end{bmatrix}$. Then $\pi(s)I_{\mathcal{K}_1} = \begin{bmatrix} 0 & 0 \end{bmatrix}$, which represents the probability of being at state groups X_1 and X_3 after executing s. On the other hand, after observing $o = aa$, the state distribution is $\tilde{\pi}(o) = \begin{bmatrix} 0 & 4/21 & 8/21 & 9/21 \end{bmatrix}$. Then $\tilde{\pi}(o)I_{\mathcal{K}_1} = \begin{bmatrix} 0 & 9/21 \end{bmatrix}$, which represents the probability of being at state groups X_1 and X_3 after observing o. Finally, $\tilde{\pi}(o)I_{\mathcal{K}_2} = \begin{bmatrix} 12/21 \end{bmatrix}$, which represents the probability of being at state group X_2 after observing o. ■

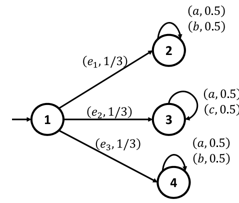

Fig. 3. A stochastic discrete-event system, with $M(\sigma) = \epsilon$ for $\sigma = e_1, e_2, e_3$, and $M(\sigma) = \sigma$ for $\sigma = a, b, c$.

B. P-Detectability

The following definition concerns the concept of Partition-based Detectability, or P-Detectability, where we use $|\cdot|_+$ to denote the number of nonzero elements in a vector.

Definition 1: Given a stochastic DES $G = (X, \Sigma, \alpha, x_0)$, an observation mask M, a partition $X_1, \ldots, X_K$ of the state space, and two sets $\mathcal{K}_1 \subseteq \mathcal{K}_2$, $(G, M, \mathcal{K}_1, \mathcal{K}_2)$ is said to be *Partition-based Detectable*, or *P-Detectable*, if

$$(\forall \tau > 0)(\exists n \in \mathbb{N}) Pr\big(s \in L(G): |s| \geq n, |\pi(s)I_{\mathcal{K}_1}|_+ > 0, \\ |\tilde{\pi}(M(s))I_{\mathcal{K}_2}|_+ > 1\big) < \tau.$$

In other words, the P-Detectability requires that any state group in $\mathcal{K}_1$ can be surely and uniquely detected with respect to state groups in $\mathcal{K}_2$ as the system evolves longer. Therefore, $\mathcal{K}_1$ denotes the set of state groups that are considered as "public", while $\mathcal{K}_2$ denotes the set of state groups that are considered as "cover" whose ambiguity with states in $\mathcal{K}_1$ should be avoided. By defining detectability using state groups of system state, rather than system state itself, the ambiguity among system states that belong to the same state group is allowed. Moreover, any ambiguity with respect to states that does not belong to any state group in $\mathcal{K}_2$ is also allowed.

Example 3: Consider the system presented in Fig. 1. Suppose one is not interested in differentiating between "Lane Keeping" and "Lane Change". Then one can partition the system state into $X_1 = \{\text{Manual Control}\}$, $X_2 = \{\text{Lane Keeping, Lane Change}\}$, and $X_3 = \{\text{ACC Only}\}$. In addition, let $\mathcal{K}_1 = \mathcal{K}_2 = \{1,2,3\}$. Then the proposed P-Detectability notion can be used to verify if the system setup allows detecting systems operation mode while ignoring the ambiguity between "Lane Keeping" and "Lane Change". ■

Example 4: Consider the system in Fig. 2. Let's consider different scenarios. (1) Let $X_1 = \{1\}$, $X_2 = \{2\}$, $X_3 = \{3\}$, $X_4 = \{4\}$, $\mathcal{K}_1 = \mathcal{K}_2 = \{2,3,4\}$. Then the system is not P-Detectable since X_2 and X_3 are ambiguous with non-decreasing probability once system enters them. (2) Let $X_1 = \{1\}$, $X_2 = \{2,3\}$, $X_3 = \{4\}$, $\mathcal{K}_1 = \mathcal{K}_2 = \{2,3\}$. Then the system is P-Detectable since now the ambiguity between "2" and "3" is allowed. (3) Finally, let $X_1 = \{1\}$, $X_2 = \{2\}$, $X_3 = \{3\}$, $X_4 = \{4\}$, $\mathcal{K}_1 = \{2\}$, $\mathcal{K}_2 = \{2,4\}$. Then the system is P-Detectable. This is because, after executing e_1 and landing at state "2", the only way to keep state "2" ambiguous from its cover (i.e., state "4") is to execute traces that does not contain event b, whose probability monotonically decreases as system evolves. ■

Example 5: Consider another system as shown in Fig. 3, where $M(\sigma) = \epsilon$ for $\sigma = e_1, e_2, e_3$, and $M(\sigma) = \sigma$ for

$\sigma = a, b, c$. Let's consider four different scenarios. (1) Let $X_1 = \{1\}$, $X_2 = \{2\}$, $X_3 = \{3\}$, $X_4 = \{4\}$, $\mathcal{K}_1 = \mathcal{K}_2 = \{2, 3, 4\}$. Then the system is not P-Detectable due to the fact that ambiguity between state "2" and "4" occurs with non-decreasing probability. (2) Let $X_1 = \{1\}$, $X_2 = \{2\}$, $X_3 = \{3\}$, $X_4 = \{4\}$, $\mathcal{K}_1 = \mathcal{K}_2 = \{2, 3\}$. Then the system is P-Detectable. For traces that lead the system entering state "2" or "3", after observing the system long enough, the probability of not detecting current system state surely converges to 0. (3) Let $X_1 = \{1\}$, $X_2 = \{2\}$, $X_3 = \{3\}$, $X_4 = \{4\}$, $\mathcal{K}_1 = \{2\}$, $\mathcal{K}_2 = \{2, 3\}$. Then the system is still P-Detectable. (4) Finally, let $X_1 = \{1\}$, $X_2 = \{2, 4\}$, $X_3 = \{3\}$, $\mathcal{K}_1 = \mathcal{K}_2 = \{2, 3\}$. Then the system is P-Detectable since now states "2" and "4" belong to the same state group X_2 and their ambiguity is allowed. ■

C. Comparison to A-Detectability and A-Diagnosability

The following definition concerns the concept of A-Detectablility.

Definition 2 ([21]): Given a stochastic DES $G = (X, \Sigma, \alpha, x_0)$ and an observation mask M, (G, M) is said to be A-Detectable if

$$(\forall \tau > 0)(\exists n \in \mathbb{N}) \\ Pr(s{:}s \in L(G), |s| \geq n, |\tilde{\pi}(M(s))|_+ > 1) < \tau.$$

The follow lemma shows that the proposed P-Detectability is more general than A-Detectability.

Lemma 1: Given a stochastic DES $G = (X, \Sigma, \alpha, x_0)$, let the partition of X be: $K = |X|$ and $X_k = \{x_k\}$ for all $k = 1, \ldots, K$. In addition, let $\mathcal{K}_1 = \mathcal{K}_2 = \{1, \ldots, K\}$. Then (G, M) is A-Detectable if and only if $(G, M, \mathcal{K}_1, \mathcal{K}_2)$ is P-Detectable.

Example 6: Consider the system in Fig. 2. The A-Detectability requires *each* state to be detectable from all other states. Therefore the system in Fig. 2 is not A-Detectable. However, suppose one does not need to distinguish state "2" and state "3", but rather is only interested in detecting between state "4" and state group consisting of state "2" and "3". Let $X_1 = \{1\}$, $X_2 = \{2, 3\}$, $X_3 = \{4\}$, $\mathcal{K}_1 = \mathcal{K}_2 = \{2, 3\}$. Then the system is P-Detectable. On the other hand, A-Detectability requires each state to be uniquely detectable compared to all other states. Consider the system in Fig. 3, which is not A-Detectable. However, suppose we are not interested in any ambiguity with respect to state "4". Let $X_1 = \{1\}$, $X_2 = \{2\}$, $X_3 = \{3\}$, $X_4 = \{4\}$, $\mathcal{K}_1 = \mathcal{K}_2 = \{2, 3\}$. Then the system is P-Detectable. ■

The following definition and lemma concern the concept of A-Diagnosability and establish that the proposed P-Detectability is more general than A-Diagnosability.

Definition 3 ([4]): Given a stochastic DES $G = (X, \Sigma, \alpha, x_0)$ with an observation mask M, whose state space is partitioned into faulty state X_F and nonfaulty state X_N. Let the set of traces that lead G into a nonfaulty state in X_N as K. Then (G, M) is said to be A-Diagnosable, if

$$(\forall \tau > 0)(\exists n \in \mathbb{N})(\forall s \in L(G) - K) \\ Pr(st{:}t \in L(G)\backslash s, |t| \geq n, Pr_{\text{amb}}(st) > 0) < \tau,$$

where $Pr_{\text{amb}}{:}L(G) - K \rightarrow [0, 1]$ is given by $Pr_{\text{amb}}(s) = Pr(u \in K{:}M(u) = M(s))/Pr(u \in L(G){:}M(u) = M(s))$.

Lemma 2: Given a stochastic DES $G = (X, \Sigma, \alpha, x_0)$, let the partition of X be: $X_1 = X_F$ and $X_2 = X_N$. In addition, let $\mathcal{K}_1 = \{1\}$ and $\mathcal{K}_2 = \{1, 2\}$. Then (G, M) is A-Diagnosable if and only if $(G, M, \mathcal{K}_1, \mathcal{K}_2)$ is P-Detectable.

IV. Verification of P-Detectability

A. A Necessary and Sufficient Condition

Define $L(x, X_k) = \{s | \exists x' \in X_k, \alpha(x, s, x') > 0\}$, i.e., $L(x, X_k)$ is the set of traces starting from x and ending in one of the states in X_k. Further denote $L_M(x, X_k) = M(L(x, X_k))$. Given $x_1, x_2 \in X$, if there exist $s_1, s_2 \in L(G)$, such that $M(s_1) = M(s_2)$ and $\alpha(x_0, s_1, x_1) > 0$ and $\alpha(x_0, s_2, x_2) > 0$, then we say x_1 and x_2 can be ambiguously reached.

Theorem 1: Given a stochastic DES $G = (X, \Sigma, \alpha, x_0)$, an observation mask M, a partition $X_1, \ldots, X_K$ of state space, and two sets $\mathcal{K}_1 \subseteq \mathcal{K}_2$, $(G, M, \mathcal{K}_1, \mathcal{K}_2)$ is not P-Detectable if and only if there exists a recurrent state $x_1 \in \cup_{k \in \mathcal{K}_1} X_k$ such that

1) it can be ambiguously reached with another state $x_2 \in \cup_{k \in \mathcal{K}_2} X_k - X_{\text{id}(x_1)}$, i.e., x_1 and x_2 can be ambiguously reached and do not belong to the same state group. Denote the set of all such x_2 as $\mathcal{X}_2$. And
2) the generated masked language starting from x_1 and reaching at state group $X_{\text{id}(x_1)}$ is a subset of the masked language generated from $\mathcal{X}_2$ and reaching at a state of $\cup_{k \in \mathcal{K}_2} X_k - X_{\text{id}(x_1)}$, i.e., $M(L(x_1, X_{\text{id}(x_1)})) \subseteq \cup_{x_2 \in \mathcal{X}_2} M(L(x_2, \cup_{k \in \mathcal{K}_2} X_k - X_{\text{id}(x_1)}))$.

Proof: When the above condition holds, then there exists $s \in L(G)$, after executing which G reaches x_1 with $\text{id}(x_1) \in \mathcal{K}_1$. Therefore, $|\pi(s) I_{\mathcal{K}_1}|_+ > 0$. Furthermore, since x_1 is recurrent and $M(L(x_1, X_{\text{id}(x_1)})) \subseteq \cup_{x_2 \in \mathcal{X}_2} M(L(x_2, \cup_{k \in \mathcal{K}_2} X_k - X_{\text{id}(x_1)}))$, all extensions of s are ambiguous with another trace leading to $\cup_{k \in \mathcal{K}_2} X_k - X_{\text{id}(x_1)}$. Therefore, $\forall t \in L\backslash s$, $|\tilde{\pi}(M(st)) I_{\mathcal{K}_2}|_+ > 1$. According to Definition 1, the system is not P-Detectable.

When the above condition does not hold, then for all traces $s \in L(G)$ that leads to a recurrent state $x_1 \in \cup_{k \in \mathcal{K}_1} X_k$, either Condition 1) does not hold or Condition 2) does not hold. In the former case, $|\tilde{\pi}(M(s)) I_{\mathcal{K}_2}|_+ = 1$. In the latter case, let $t_1 \in L\backslash s$ be the shortest extension such that $M(t_1) \not\in \cup_{x_2 \in \mathcal{X}_2} M(L(x_2, \cup_{k \in \mathcal{K}_2} X_k - X_{\text{id}(x_1)}))$. Then, $|\tilde{\pi}(M(st_1)) I_{\mathcal{K}_2}|_+ \leq 1$. Denote the probability of t_1 as p_1. Since the conditions in Theorem 1 do not hold, then for all other extensions t of s, either $|\tilde{\pi}(M(st)) I_{\mathcal{K}_2}|_+ = 1$ or there exists an extension $t_2 \in L\backslash st$ such that $|\tilde{\pi}(M(stt_2)) I_{\mathcal{K}_2}|_+ \leq 1$. Denote the probability of t_2 as p_2. Let n_k be the length of the kth shortest unambiguous extension of s. Then we have

$$\begin{aligned}
&Pr\big(st{:}st \in L(G), |t| \geq n_k, |\tilde{\pi}(M(st)) I_{\mathcal{K}_2}|_+ > 1\big) \\
&\leq \prod_{i=1}^{k} Pr(t_i{:}t_i \in L(G)\backslash st_1 \ldots t_{i-1}, |t_i| = n_i - n_{i-1}, \\
&\qquad\qquad |\tilde{\pi}(M(st_1 \ldots t_i)) I_{\mathcal{K}_2}|_+ > 1) \\
&\quad \times Pr(t \in L(G)\backslash st_1 \ldots t_k, \\
&\qquad\qquad |\tilde{\pi}(M(st_1 \ldots t_k t)) I_{\mathcal{K}_2}|_+ > 1) \times Pr(s)
\end{aligned}$$

$$\leq \prod_{i=1}^{k}(1-p_i) \times Pr(t \in L(G)\backslash st_1 \ldots t_k,$$

$$|\tilde{\pi}(M(st_1 \ldots t_k t))I_{\mathcal{K}_2}|_+ > 1) \times Pr(s)$$

$$\leq \prod_{i=1}^{k}(1-p_i),$$

which approaches 0 when k increases (or equivalently when n_k increases), since $1-p_i < 1$ for all i. Therefore, for any $\tau > 0$, there should exist $n > 0$, such that $Pr(st{:}st \in L(G), |t| \geq n, |\tilde{\pi}(M(st))I_{\mathcal{K}_2}|_+ > 1) < \tau$. Note that the above analysis holds for any trace $s \in L(G)$ that leads to a recurrent state $x_1 \in \cup_{k\in\mathcal{K}_1} X_k$. Therefore, the system is P-Detectable, according to Definition 1. ■

Example 7: Consider the same scenarios discussed in Example 4. (1) Let $X_1 = \{1\}$, $X_2 = \{2\}$, $X_3 = \{3\}$, $X_4 = \{4\}$, $\mathcal{K}_1 = \mathcal{K}_2 = \{2, 3, 4\}$. Since "2" and "3" can be ambiguously reached and $M(L(2, X_2)) \subseteq M(L(3, X_3))$, the system is not P-Detectable. (2) Let $X_1 = \{1\}$, $X_2 = \{2, 3\}$, $X_3 = \{4\}$, $\mathcal{K}_1 = \mathcal{K}_2 = \{2, 3\}$. Though "2" and "3" can be ambiguously reached with "4", we have $M(L(2, X_2)) \nsubseteq M(L(4, X_3))$ and $M(L(3, X_2)) \nsubseteq M(L(4, X_3))$. Therefore, the system is P-Detectable. (3) Finally, let $X_1 = \{1\}$, $X_2 = \{2\}$, $X_3 = \{3\}$, $X_4 = \{4\}$, $\mathcal{K}_1 = \{2\}$, $\mathcal{K}_2 = \{2, 4\}$. Though '2" can be ambiguously reached with "4", we have $M(L(2, X_2)) \nsubseteq M(L(4, X_4))$. Therefore, the system is P-Detectable. Note that the above conclusions are same as Example 4. ■

B. Testing Algorithm

Given a stochastic DES $G = (X, \Sigma, \alpha, x_0)$ and an observation mask M, construct a current state observer $O = (Z, \Delta, \alpha_{obs}, Z_0)$ such that $Z \subseteq 2^X$. In other words, each state of O is a subset X. The steps to construct O is as follows.

- The initial state $Z_0 = \{x \in X | \exists s \in L, M(s) = \epsilon, \alpha(x_0, L, x) > 0\}$;
- For any $z \in Z$ and $\delta \in \Delta$, $\alpha_{obs}(z, \delta) = \{x \in X | \exists x' \in z, \exists s \in \Sigma^*, M(s) = \delta, \alpha(x', s, x) > 0\}$.

Next, construct a testing automaton $T = (X \times Z, \sigma, \alpha_t, (x_0, Z_0))$ such that for $(x_1, z_1) \in X \times Z$, $(x_2, z_2) \in X \times Z$, and $\sigma \in \Sigma$, $\alpha_t((x_1, z_1), \sigma, (x_2, z_2)) = \alpha(x_1, \sigma, x_2)$ if $[(M(\sigma) = \epsilon) \wedge (z_1 = z_2)] \vee [(M(\sigma) \in \Delta) \wedge z_2 = \alpha_{obs}(z_1, M(\sigma))]$, and otherwise $\alpha_t((x_1, z_1), \sigma, (x_2, z_2)) = 0$. In other words, the testing automaton T is a composition of the system G with its observer O. Each state in T consists of two coordinates, $x \in X$ and $z \in Z$. The first coordinate evolves according to the dynamics of G and the second coordinate includes the corresponding current state estimate. It is trivial to see that T generates the same language as G, i.e., $L(T) = L(G)$.

Theorem 2: Given a stochastic DES $G = (X, \Sigma, \alpha, x_0)$, an observation mask M, a partition $X_1, \ldots, X_K$ of state space, and two sets $\mathcal{K}_1 \subseteq \mathcal{K}_2$, $(G, M, \mathcal{K}_1, \mathcal{K}_2)$ is not P-Detectable if and only if there exists a recurrent state $(x_1, \bar{z})$ of T such that $x_1 \in \cup_{k\in\mathcal{K}_1} X_k$ and there exists $x_2 \in \bar{z}$ such that $x_2 \in \cup_{k\in\mathcal{K}_2} X_k - X_{\mathrm{id}(x_1)}$.

Proof: When the above condition holds, then x_1 and x_2 can be ambiguously reached. Since $(x_1, \bar{z})$ is recurrent, the generated language from x_1 is a subset of the generated language from x_2, as otherwise $(x_1, \bar{z})$ will be a transient state. According to Theorem 1, the system is not P-Detectable.

On the other hand, when the above condition does not hold, then if there exists a state $(x_1, \bar{z})$ of T such that $x_1 \in \cup_{k\in\mathcal{K}_1} X_k$ and there exists $x_2 \in \bar{z}$ such that $x_2 \in \cup_{k\in\mathcal{K}_2} X_k - X_{\mathrm{id}(x_1)}$, $(x_1, \bar{z})$ must be transient. This implies that the generated language from x_1 is a not a subset of the generated language from x_2. According to Theorem 1, the system is P-Detectable. ■

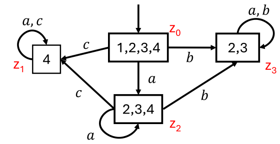

Fig. 4. State observer for system in Fig. 2.

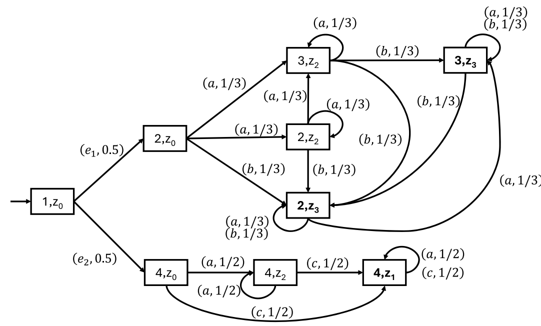

Fig. 5. Testing automaton for system in Fig. 2.

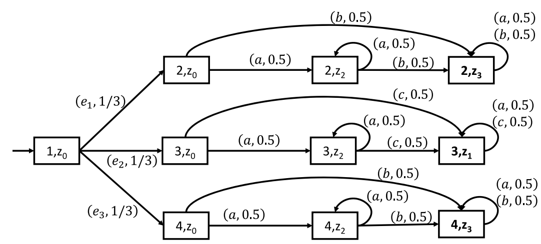

Fig. 6. Testing automaton for system in Fig. 3.

Example 8: Consider the system in Example 4 and Fig. 2. The corresponding state observer O and testing automaton T are plotted in Fig. 4 and Fig. 5. There are three recurrent states in T, namely, $(2, z_3)$, $(3, z_3)$, and $(4, z_1)$, as marked in bold in Fig. 5. Let $X_1 = \{1\}$, $X_2 = \{2\}$, $X_3 = \{3\}$ and $X_4 = \{4\}$. When $\mathcal{K}_1 = \{2\}$ and $\mathcal{K}_2 = \{2, 4\}$, the system is P-Detectable since for the recurrent state $(2, \bar{z})$ of T, $\bar{z} \cap X_4 = \emptyset$. When $\mathcal{K}_1 = \{2\}$ and $\mathcal{K}_2 = \{2, 3, 4\}$, the system is not P-Detectable as the recurrent state $(2, z_3)$ satisfies the condition in Theorem 2. ■

Example 9: Consider the system in Example 5 and Fig. 3. The corresponding testing automaton T is plotted in Fig. 6, where $z_0 = \{1, 2, 3, 4\}$, $z_1 = \{3\}$, $z_2 = \{2, 3, 4\}$, and $z_3 = \{2, 4\}$. There are three recurrent states in T, namely, $(2, z_3)$, $(3, z_1)$, and $(4, z_3)$, as marked in bold in Fig. 6. When $X_1 = \{1\}$, $X_2 = \{2\}$, $X_3 = \{3\}$, $X_4 = \{4\}$, $\mathcal{K}_1 = \mathcal{K}_2 = \{2, 3, 4\}$, the system is not P-Detectable as both recurrent states $(2, z_3)$ and $(4, z_3)$ satisfy the condition in Theorem 2. When $X_1 = \{1\}$, $X_2 = \{2\}$, $X_3 = \{3\}$, $X_4 = \{4\}$, $\mathcal{K}_1 = \mathcal{K}_2 = \{2, 3\}$, the system is P-Detectable. This is because for $(2, z_3)$, $z_3 \cap X_3 = \emptyset$ and for $(3, z_1)$, $z_1 \cap X_2 = \emptyset$. When $X_1 = \{1\}$, $X_2 = \{2, 4\}$, $X_3 = \{3\}$, $\mathcal{K}_1 = \mathcal{K}_2 = \{2, 3\}$, the system is P-Detectable. In this case, for both $(2, z_3)$ and $(4, z_3)$, $z_3 \cap X_3 = \emptyset$. Note that the above conclusions are the same as Example 5. ■

Remark 1: Unfortunately, verification of the conditions in Theorem 1 and Theorem 2 requires exponential complexity. In fact, according to Lemmas 1 and 2, the verifications of A-Detectability and A-Diagnosability can be reduced to the verification of P-Detectability. As shown in [6], [21], verifications of A-Detectability and A-Diagnosability are PSPACE-Hard. Therefore, the verification of P-Detectability is also PSPACE-Hard. The exploration of specific types of partitions that could reduce the verification complexity remains a future work direction.

V. Conclusion

This letter introduces a new notion called *Partition-based Detectability*, or *P-Detectability*, for analyzing detectability in stochastic discrete-event systems based on partitions of the system state space. The proposed notion is defined over a selected subset of state groups, focusing on the system capability to detect certain state group from other state groups while ignoring the ambiguity between individual states within the same state group. The proposed P-Detectability allows users to define customized public and cover sets. Compared to existing notions such as A-Detectability and A-Diagnosability, P-Detectability is shown to be a more general framework. A necessary and sufficient condition for verifying P-Detectability, along with an algorithm for practical testing, are presented. Several illustrative examples are included to demonstrate the key concepts. As the testing algorithm requires exponential complexity, future work will focus on reducing the verification complexity using probabilistic testing algorithm [25]. Moreover, as the partition of states is determined by the system designer and will influence the verification, future work will focus on how to optimize state partition while keeping *essential* information detectable.

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